

Quantum Scales of Galaxies from Self-interacting Ultralight Dark Matter

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Abstract We derive the characteristic scales for physical quantities of dwarf galaxies, such as mass, size, acceleration, and angular momentum, within the self-interacting ultralight dark matter (ULDM) model. Due to the small mass of ULDM, even minor self-interactions can drastically alter these scales in the Thomas-Fermi limit. We suggest that these characteristic scales are connected to mysteries of observed galaxies. Oscillation of ULDM field can explain the current cosmological density of dark matter. Many cosmological constraints suggest that the energy scale $m\tilde{}$ for self-interacting ULDM is typically of the order 10 eV , whereas the mass m for the non-interacting case is around 10^{-21} eV . Self-interacting ULDM provides the better explanation for cosmological observations than the non-interacting case.

I. INTRODUCTION

The ultralight dark matter (ULDM) model has emerged as a compelling alternative to cold dark matter (CDM), in which dark matter particles have an exceptionally small mass m , typically on the order of $10\text{--}22\text{ eV}$, and exist in a Bose-Einstein condensate (BEC) state. (For reviews, see [1–6]). This model is known by various other names, including fuzzy DM, BEC DM, scalar field DM, ultra-light axion, and wave- ψ DM [7–24]. The long de Broglie wave length $\lambda_{\text{dB}} = h/mv$ kpc of ULDM determines the typical size of galaxies, where v is the typical velocity of the halo dark matter. This scale can give the size and mass of the smallest galaxies [25, 26] and resolve the small scale issues of CDM including the core-cusp problem, the satellite galaxy plane problem and the missing satellite problem [27–31]. ULDM has also been proposed to address the mysteries of black holes, including the M-sigma relation [32] and the final parsec problem [33, 34]. In Ref. [35], the characteristic scales of physical properties of galaxies, such as angular momentum and mass, in the fuzzy DM model (i.e., non-interacting ULDM) were studied and found to be consistent with observations. In this model, quantum pressure arising from the uncertainty principle counteracts the gravitational force.

In the ULDM model, dark matter halos of galaxies consist of central cores (solitons) and soliton-like granules surrounding them. Note that, in this work, the typical scales apply to these cores and granules of large galaxies or to cores of dwarf galaxies, but not to galaxies as a whole. These characteristic scales are useful for studying galaxy evolution, particularly in dwarf galaxies, which are strongly dominated by dark matter. One can use these scales to make order-of-magnitude estimates that illuminate the fundamental physics of galaxies. We emphasize that this model addresses the non-baryonic component of dark matter. Observational data, such as those from the Planck Collaboration, indicate that the total dark matter content in the Universe includes both baryonic and non-baryonic components [36]. Our model does not aim to account for the baryonic dark matter contribution.

Fuzzy dark matter (FDM), however, encounters observational challenges, most notably from the Lyman-alpha forest data [37–39], which implies $m \sim 10\text{--}21\text{ eV}$ [40]. Incorporating self-interactions into the ULDM framework [9, 23, 41] has been suggested as a way to alleviate these tensions [42], accommodating a wider mass range that aligns with cosmological observations [43, 44]. More recently, self-interacting ULDM has also been proposed as a possible solution to the Hubble tension [45], neutrino mass and the electroweak scale problems [46].

In this paper, we derive typical scales for physical quantities of galaxies in self-interacting ULDM model, which can be related to mysteries of galaxies. We compare the characteristic scales predicted by the two models. In Section II, we review the Jeans length of generic ULDM. In Section III, the characteristic scales are derived for fuzzy DM. In Section IV, we consider the self-interacting case. In Section V, we discuss the results and outlook.

II. JEANS LENGTH OF ULTRALIGHT DARK MATTER

In this section, we review the basics of ULDM. The ULDM field can be a scalar field ϕ with an action

$$S = \int \sqrt{-g} d^4x \left[\frac{-R}{16\pi G} - \frac{g^{\mu\nu}}{2} \phi_{;\mu}^* \phi_{;\nu} - U(\phi) \right], \quad (1)$$

where the potential for the field can be

$$U(\phi) = \frac{m^2 c^2}{2\hbar^2} |\phi|^2 + \frac{\lambda |\phi|^4}{4\hbar c} = \frac{m^2 c^2}{2\hbar^2} |\phi|^2 + \frac{2\pi a_s m}{\hbar^2} |\phi|^4. \quad (2)$$

Here, $\lambda = 8\pi a_s m c / \hbar$ is a dimensionless coupling and $a_s = \lambda \hbar / (8\pi m c)$ is a scattering length [4]. FDM corresponds to the case $\lambda = 0$. In the Newtonian limit, odd-power terms can be ignored because they average out to zero over galactic time scales as the field rapidly oscillates with a frequency of $O(m)$. We adopt the quartic term with $\lambda > 0$, which is the highest even power term that remains renormalizable. We do not consider the extension to cosine potential, which gives an effective attractive quartic term [47]. The evolution of the field is described by the following equation

$$\square \phi + 2 \frac{dU}{d|\phi|^2} \phi = 0, \quad (3)$$

where $\square$ is the d'Alembertian. Since galaxies are non-relativistic, it is useful to define ψ as

$$\phi(t, \mathbf{x}) = \frac{1}{\sqrt{2m}} \left[e^{-imt} \psi(t, \mathbf{x}) + e^{imt} \psi^*(t, \mathbf{x}) \right]. \quad (4)$$

Then the field amplitude becomes $|\phi|^2 = \frac{\hbar^2}{m^2} |\psi|^2$.

In the Newtonian limit, the macroscopic wave function ψ satisfies the following nonlinear Schrödinger-Poisson equation (SPE);

$$\begin{aligned} i\hbar \partial_t \psi &= -\frac{\hbar^2}{2m} \nabla^2 \psi + mV\psi + \frac{\lambda \hbar^3}{2cm^3} |\psi|^2 \psi, \\ \nabla^2 V &= 4\pi G \rho, \end{aligned} \quad (5)$$

where the DM mass density $\rho = m|\psi|^2$, and V is the gravitational potential.

Cosmological structure formation is described by an equation for the density contrast $\delta \equiv \delta\rho/\bar{\rho} = (\rho - \bar{\rho})/\bar{\rho} = \sum_k \delta_k e^{ik \cdot r}$ with a wave vector k of the perturbation [48],

$$\frac{d^2 \delta_k}{dt^2} + [(c_q^2 + c_s^2)k^2 - 4\pi G \bar{\rho}] \delta_k = 0, \quad (6)$$

where $\bar{\rho}$ is the cosmological background dark matter density, $c_q = \hbar k / 2m$ is a quantum velocity and $c_s = \sqrt{4\pi a_s \hbar^2 \bar{\rho} / m^3} = \sqrt{\hbar^3 \lambda \bar{\rho} / 2cm^4}$ is the sound velocity from self-interaction. The Jeans length corresponds to the wave vector k_J satisfying $(c_q^2 + c_s^2)k^2 - 4\pi G \bar{\rho} = 0$. The DM system becomes unstable to perturbations with $k < k_J$, resulting in the formation of cosmic structures.

During the structure formation, gravity dominates over pressure, and we have the equation: $\frac{d^2 \delta_k}{dt^2} \simeq 4\pi G \bar{\rho} \delta_k$. This leads to a typical time scale for structure formation [4]:

$$t_c = \frac{1}{\sqrt{G \bar{\rho}}}, \quad (7)$$

which is approximately the order of the Hubble time at the time of structure formation. Therefore, t_c depends on the redshift at which individual galaxies form. However, this does not mean that all galaxies have the same time scale. In fact, t_c represents the time scale of the lightest galaxy or cores of heavy dark matter halos formed at a given redshift. This time scale can be further justified by the following argument. An astronomical object like a dwarf galaxy with a typical mass M and radius R has a characteristic time scale given by: $\frac{R}{\sqrt{GM/R}} \simeq \frac{1}{\sqrt{G\rho_c}}$, where $\rho_c = O(10^2) \bar{\rho}$ is the typical density of the object. Therefore, Eq. (7) provides a reasonable estimate for the characteristic time scale in galactic dynamics.

III. QUANTUM SCALES OF FUZZY DARK MATTER

In this section we review the case of the free field model ($\lambda = 0$), i.e., FDM model [35]. From the condition $c_q^2 k^2 = 4\pi G \bar{\rho}$, one can obtain the quantum Jeans length scale [6, 47] at a redshift z :

$$\lambda_Q(z) = \frac{2\pi}{k} = \left(\frac{\pi^3 \hbar^2}{G m^2 \bar{\rho}(z)} \right)^{1/4} = 71.75 \text{ kpc} \left(\frac{m}{10^{-22} \text{ eV}} \right)^{-1/2} \left(\frac{\bar{\rho}}{10^{-7} M_\odot / \text{pc}^3} \right)^{-1/4} \propto (1+z)^{-3/4}, \quad (8)$$

where $m_{22} = m/10^{-22} \text{ eV}$.

A different length scale, characterizing a stable galactic core, emerges from the equilibrium between the self-gravitational force of a ULDM soliton with a mass M and quantum pressure;

$$R_{99} = 9.95 \left(\frac{\hbar}{m} \right)^2 \frac{1}{GM} = 8.5 \text{ kpc} \left(\frac{10^{-22} \text{ eV}}{m} \right)^2 \frac{10^8 M_\odot}{M}, \quad (9)$$

where R_{99} is the radius containing 99% of the ULDM mass [49]. We choose R_{99} as a length scale $\mathbf{x}_c$ for FDM, which is of the order of the de Broglie wavelength of the ULDM particles in the soliton. R_{99} is inversely proportional to the soliton mass, which is not typical of observed galactic cores.

The scales of ULDM systems are determined by two mass parameters m and M . From λ_Q the quantum Jeans mass can be derived as

$$M_Q(z) = \frac{4\pi}{3} \left(\frac{\lambda_Q}{2} \right)^3 \bar{\rho} = \frac{\pi^{13/4}}{6} \left(\frac{\hbar}{G^{1/2} m} \right)^{3/2} \bar{\rho}(z)^{1/4} = 1.93 \times 10^7 M_\odot \left(\frac{m}{10^{-22} \text{ eV}} \right)^{-3/2} \left(\frac{\bar{\rho}}{10^{-7} M_\odot / \text{pc}^3} \right)^{1/4} \propto (1+z)^{3/4}, \quad (10)$$

which is the typical mass scale of dwarf galaxies or galactic cores and appropriate for M . This does not imply that all galaxies have identical masses. Simulations suggest that the core mass scales as $M_h^{1/3}$, where M_h is the total halo mass [50].

Inserting $M = M_Q$ into Eq. (9) gives

$$\mathbf{x}_c = R_{99} = \frac{1.44 \sqrt{\frac{\hbar}{m}}}{(G\bar{\rho})^{1/4}} = 43.97 \text{ kpc} \left(\frac{m}{10^{-22} \text{ eV}} \right)^{-1/2} \left(\frac{\bar{\rho}}{10^{-7} M_\odot / \text{pc}^3} \right)^{-1/4}, \quad (11)$$

which is $0.61 \lambda_Q$. R_{99} is proportional to λ_Q but slightly smaller than λ_Q . (See Fig. 1)

From these fundamental typical scales and Eq. (10) one can easily obtain other physical scales. The typical acceleration scale is given by

$$a_c = \mathbf{x}_c / t_c^2 = 0.0044 G^3 m^4 M^3 / \hbar^4 = 8.38 \times 10^{-14} \text{ meter/s}^2 \left(\frac{m}{10^{-22} \text{ eV}} \right)^4 \left(\frac{M}{10^8 M_\odot} \right)^3 \simeq \sqrt{\frac{\hbar}{m}} (G\bar{\rho})^{3/4}, \quad (12)$$

which is similar to the Modified Newtonian dynamics (MOND) scale $a_0 = 1.2 \times 10^{-10} \text{ meter/s}^2$ for $m \simeq 10^{-21} \text{ eV}$. In Ref. [51], it is suggested that MOND arises as an effective phenomenon of ULDM, with a_c linked to the observed radial acceleration relation, if galactic cores have a similar mass M . In that scenario ULDM halos have a solitonic core and granules interacting with baryons. At a specific radius, the total density of dark matter and baryons follows an isothermal profile, where the acceleration matches a_c , possibly explaining the empirical radial acceleration relation. However, precise numerical analysis is needed to validate this scenario.

Using $\bar{\rho} = (1296 G^3 m^6 M^4) / (\hbar^6 \pi^{13})$ from Eq. (10) one can obtain the typical velocity

$$v_c \equiv \mathbf{x}_c / t_c = 0.21 GMm / \hbar = 4.71 \text{ km/s} \left(\frac{M}{10^8 M_\odot} \right) \left(\frac{m}{10^{-22} \text{ eV}} \right) \simeq \sqrt{\frac{\hbar}{m}} (G\bar{\rho})^{1/4}, \quad (13)$$

which corresponds to a typical velocity in dwarf galaxies and leads to a typical angular momentum of galactic halos

$$L_c = M \mathbf{x}_c v_c = 2.09 \hbar \frac{M}{m} = 2.33 \times 10^{96} \hbar \left(\frac{M}{10^8 M_\odot} \right) \left(\frac{10^{-22} \text{ eV}}{m} \right) \simeq \frac{(\frac{\hbar}{m})^{5/2} \bar{\rho}^{1/4}}{G^{3/4}}. \quad (14)$$

This implies that the typical angular momentum scales with the number of particles times $\hbar$, which clearly reveals the quantum nature of the model. The typical density is

$$\rho_c \equiv M / \mathbf{x}_c^3 = 0.001 \frac{G^3 m^6 M^4}{\hbar^6} = 1.62 \times 10^{-4} M_\odot / \text{pc}^3 \left(\frac{m}{10^{-22} \text{ eV}} \right)^6 \left(\frac{M}{10^8 M_\odot} \right)^4 \propto (1+z)^3, \quad (15)$$

which is independent of m and implies a high density at a high redshift. This could facilitate the early formation of supermassive black holes [52].

The typical surface density $\Sigma \equiv M/x_c^2$ is

$$\Sigma = 0.01 \frac{G^2 m^4 M^3}{\hbar^4} = 1.38 \, M_\odot / pc^2 \left(\frac{m}{10^{-22} eV} \right)^4 \left(\frac{M}{10^8 M_\odot} \right)^4 \propto (1+z)^3 \quad (16)$$

which fails to explain the observed data [53, 54]. Since Eq. (9) predicts that smaller cores are more massive and exhibit higher values of Σ , this contradicts observations indicating that Σ remains approximately constant at $75 M_\odot / pc^2$.

The scale for the gravitational potential,

$$V_c = \frac{GM}{x_c} = \frac{0.1 \, m^2}{\hbar^2} (GM)^2 = 5.6 \times 10^{-10} c^2 \left(\frac{m}{10^{-22} eV} \right)^2 \left(\frac{M}{10^8 M_\odot} \right)^2, \quad (17)$$

becomes relativistic when $M \simeq 10^{12} M_\odot$ for $m = 10^{-21} eV$. This mass M is similar to the maximum galaxy mass. All of the above constraints on galactic scales appear to favor a mass of approximately $m \simeq 10^{-21} eV$.

IV. SELF-INTERACTING CASE

In this section we consider the self-interacting case with a quartic potential $U(\phi)$ in Eq. (2) with $\lambda > 0$ [9]. In the Thomas-Fermi (TF) limit, the kinetic term can be ignored and the physical quantities often depend on the single parameter $\tilde{m} \equiv m/\lambda^{1/4}$, which represents the typical energy scale of ϕ .

In this limit the equation for the density contrast δ is now

$$\frac{d^2 \delta_k}{dt^2} + [(c_s^2)k^2 - 4\pi G \bar{\rho}] \delta_k = 0. \quad (18)$$

From the equation one can obtain the Jeans length from self-interaction [55],

$$\lambda_J = 2\pi\hbar \sqrt{\frac{a_s}{Gm^3}} = 2R_{TF} = \sqrt{\frac{\pi\hbar^3\lambda}{2cGm^4}} = 0.978 \, kpc \left(\frac{\tilde{m}}{10eV} \right)^{-2}, \quad (19)$$

which does not depend on the density and time, unlike λ_Q of FDM.

In the TF limit, the exact ground state solution of SPE is given by [4, 9]

$$|\psi|^2 = \frac{|\psi(0)|^2 R_{TF}}{\pi r} \sin\left(\frac{\pi r}{R_{TF}}\right), \quad (20)$$

where the soliton size is

$$R_{TF} = \pi\hbar \sqrt{\frac{a_s}{Gm^3}} = \sqrt{\frac{\pi\hbar^3\lambda}{8cGm^4}}. \quad (21)$$

From λ_J we can obtain the Jeans mass for ULDM,

$$M_J(z) \equiv \frac{4\pi}{3} \left(\frac{\lambda_J}{2} \right)^3 \bar{\rho} = \frac{4\pi^4 \hbar^3}{3} \left(\frac{a_s}{Gm^3} \right)^{3/2} \bar{\rho} = \frac{\pi^{5/2}}{\sqrt{288}} \left(\frac{\hbar^3 \lambda}{cGm^4} \right)^{3/2} \bar{\rho} = 49 \, M_\odot \left(\frac{\tilde{m}}{10eV} \right)^{-6} \left(\frac{\bar{\rho}}{10^{-7} M_\odot / pc^3} \right) \propto (1+z)^3, \quad (22)$$

where z is the redshift when the perturbation starts to grow, and $\bar{\rho}(z) = \rho(0)(1+z)^3$. Since the present value of $\bar{\rho}$ is about $10^{-7} M_\odot / pc^3$, $M_J(z \simeq 0)$ is too small to explain the observed galaxies, if they are formed at low redshifts. This discrepancy can be resolved by assuming that the collapse of density perturbations leading to galaxy formation began at very high redshifts ($z \gg 10$), as recent James Webb observations suggest [56, 57]. To account for the early formation of high-redshift galaxies, dark matter perturbations must begin growing at significantly higher redshifts. For example, $M_J(z = 100) \simeq 5 \times 10^7 M_\odot$, which is similar to the mass of the smallest galaxies. Somewhat smaller values of $\tilde{m}$ can also help increase M_J .

The TF limit corresponds to $\lambda_J > \lambda_Q$, which implies [41]

$$\lambda > 5.3 \times 10^{-89} \left(\frac{m}{10^{-22} eV} \right)^3 \left(\frac{\bar{\rho}(z)}{10^{-7} M_\odot / pc^3} \right)^{-1/2}. \quad (23)$$

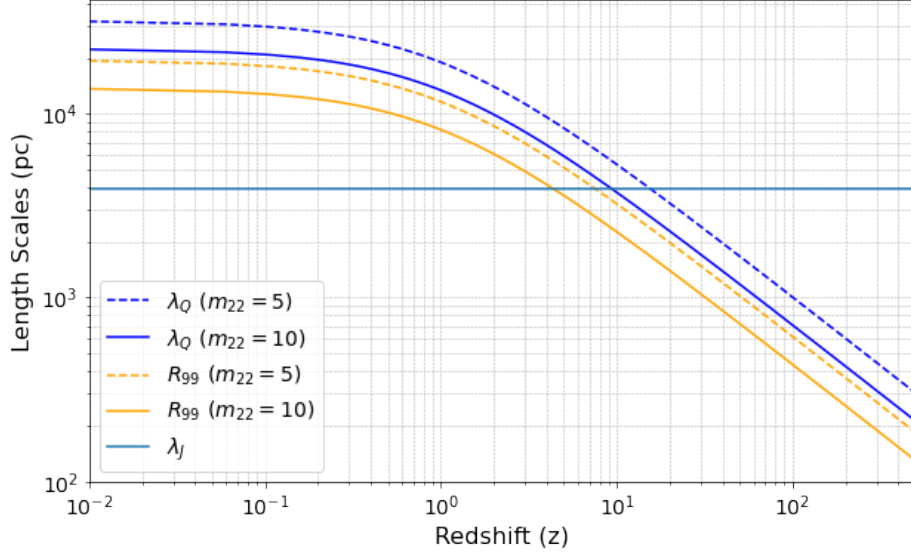


FIG. 1: The evolution of the typical length scales of galaxies versus redshift z . The upper two curves represent λ_Q for $m_{22} = m/10^{-22} \text{ eV} = 5$ and $m_{22} = 10$, respectively, while the lower two curves represent the corresponding R_{99} . The horizontal line represents $\lambda_J = 2R_{TF}$ in the TF limit with $\tilde{m} = 5 \text{ eV}$. When $\lambda_J > \lambda_Q$, the size and mass of a dark matter halo formed at that time are governed by self-interaction-induced repulsion rather than quantum pressure, especially at high redshifts $z \gg 10$ and for larger mass $m \gtrsim 10^{-21} \text{ eV}$. We expect galaxy density perturbations to begin collapsing in the redshift range $O(10) \lesssim z \lesssim O(100)$.

This corresponds to the condition that the spatial size of initial dark matter density perturbations is determined by the repulsion not by the quantum pressure. (See Fig. 2.) If we also impose the condition $R_{TF} > R_{99}$, we obtain the bound

$$\lambda > 3.02 \times 10^{-90} \left(\frac{M}{10^8 M_\odot} \right)^{-2}, \quad (24)$$

which notably does not depend on the particle mass m . This condition ensures that galactic cores are stabilized by the repulsive force arising from the quartic interaction.

For $\lambda_J > \lambda_Q$, it is natural to represent physical quantities with $M = M_J$ and $\tilde{m}$. The dynamical time scale $t_c \sim \frac{1}{\sqrt{G\rho}}$ becomes [4]

$$t_c = \left(\frac{R_{TF}^3}{GM} \right)^{1/2} = \pi^{3/4} \left(\frac{\hbar^9 \lambda^3}{512 c^3 G^5 m^{12}} \right)^{1/4} \sqrt{\frac{1}{M}} \propto (1+z)^{-3/2}. \quad (25)$$

We have the length scale $x_c = R_{TF}$, the time scale t_c , and the mass scale $M = M_J$ for the self-interacting ULDM, from which we can derive the typical scales for various other physical quantities as follows. Since x_c , t_c and M are functions of λ/m^4 , we expect many derived quantities from them to have a dependency on $\tilde{m} \equiv m/\lambda^{1/4}$.

Then, the typical density $\rho_c \equiv M/\mathbf{x}_c^3$ is

$$\rho_c = \frac{2\sqrt{2}M \left(\frac{cGm^4}{\lambda} \right)^{3/2}}{\hbar^{9/2} \pi^{3/2}} = 0.106 M_\odot / \text{pc}^3 \left(\frac{\tilde{m}}{10 \text{ eV}} \right)^6 \left(\frac{M}{10^8 M_\odot} \right) \propto (1+z)^3, \quad (26)$$

which is independent of $\tilde{m}$.

The typical velocity is

$$v_c \equiv \mathbf{x}_c/t_c = \frac{2^{7/4} \left(\frac{cG^3 m^4}{\hbar^3 \lambda} \right)^{1/4}}{\pi^{1/4}} \sqrt{M} = 59.28 \text{ km/s} \left(\frac{M}{10^8 M_\odot} \right)^{1/2} \left(\frac{\tilde{m}}{10 \text{ eV}} \right) \propto (1+z)^{3/2}, \quad (27)$$

which is similar to the typical velocity dispersion in a dwarf galaxy. v_c leads to a typical angular momentum in turn

$$L_c = M \mathbf{x}_c v_c = \left(\frac{32\pi G \hbar^3 \lambda}{cm^4} \right)^{1/4} M^{3/2} = 3.375 \times 10^{96} \hbar \left(\frac{M}{10^8 M_\odot} \right)^{3/2} \left(\frac{10 \text{ eV}}{\tilde{m}} \right) \propto (1+z)^{9/2}, \quad (28)$$

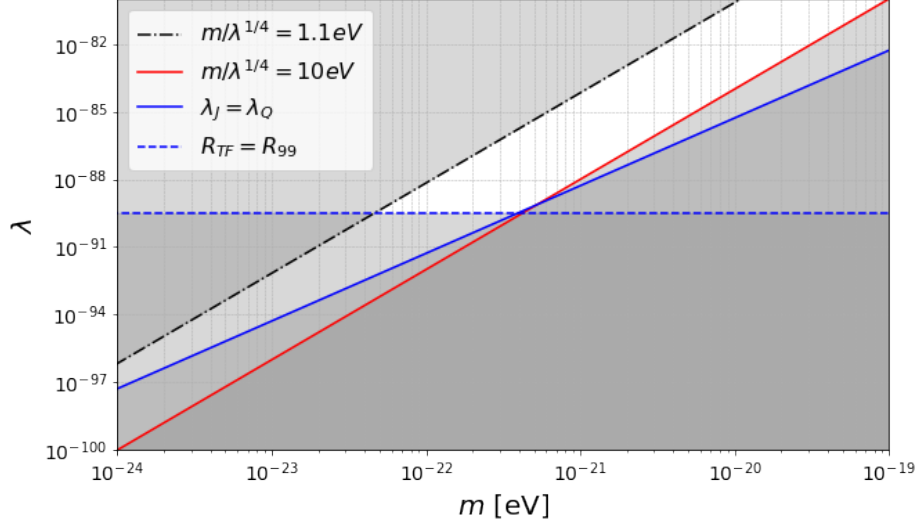


FIG. 2: Cosmological constraints on the mass m and coupling constant λ of ULDM are shown. The gray region denotes the parameter space excluded by current constraints. The dash-dot line corresponds to the bound from the dark matter density Ω_ϕ (Eq. (36)). The red solid line indicates the observational constraint from galaxies, $\tilde{m} \lesssim 10$ eV. The blue line marks the boundary where the condition $\lambda_J > \lambda_Q$ (Eq. (23)) holds at redshift $z = 100$. The dashed horizontal line represents the condition $R_{TF} > R_{99}$ for a soliton mass of $M = 10^8 M_\odot$.

which is not of the order of $\hbar(M/m)$ as in the fuzzy dark matter.

The typical acceleration scale is given by

$$a_c = \mathbf{x}_c/t_c^2 = \frac{16cG^2m^4M}{\pi\hbar^3\lambda} = 1.163 \times 10^{-10} \text{ meter}/s^2 \left(\frac{\tilde{m}}{10\text{eV}} \right)^4 \left(\frac{M}{10^8 M_\odot} \right) \propto (1+z)^3 \quad (29)$$

which is similar to the MOND scale $a_0 = 1.2 \times 10^{-10} \text{ meter}/s^2$ for $M = 10^8 M_\odot$. Galaxies with similar core mass M are expected to exhibit similar values of a_c .

The scale for the gravitational potential,

$$V_c = GM/\mathbf{x}_c = \frac{2a_s\hbar^2\pi^3\bar{\rho}}{3m^3} = \frac{GM\sqrt{\frac{2}{\pi}}}{\sqrt{\frac{\hbar^3\lambda}{cGm^4}}} = 4.888 \times 10^{-9}c^2 \left(\frac{\tilde{m}}{10 \text{ eV}} \right)^2 \left(\frac{M}{10^8 M_\odot} \right) \propto (1+z)^3, \quad (30)$$

becomes relativistic when $M \simeq 10^{16} M_\odot$ for $\tilde{m} = 10 \text{ eV}$.

The typical surface density Σ is

$$\Sigma = \frac{\hbar\pi^{3/2}\sqrt{\frac{\hbar\lambda}{cGm^4}}\bar{\rho}}{6\sqrt{2}} = 5.124 M_\odot/pc^2 \left(\frac{\tilde{m}}{10\text{eV}} \right)^{-2} \left(\frac{\bar{\rho}}{10^{-2}M_\odot/pc^3} \right) = 104.4 M_\odot/pc^2 \left(\frac{\tilde{m}}{10\text{eV}} \right)^{-2} \left(\frac{M}{10^8 M_\odot} \right) \propto (1+z)^3 \quad (31)$$

which is similar to the observed value $\Sigma \simeq 75 M_\odot/pc^2$ for $M = 10^8 M_\odot$. Of course, Σ in the TF limit is also redshift dependent. However, if the dwarf galaxies observed start forming around the same redshift, for example, $z \simeq 100$, they can exhibit a comparable Σ . The dependence of Σ on M is weaker than in the FDM case.

Considering all these factors, self-interacting ULDM offers a more compelling explanation for the observed galaxies if their cores formed at similarly high redshifts. To be conclusive, we need a high-resolution simulation of structure formation with self-interacting ULDM, as cores in a massive halo may exhibit slightly different properties than those of a single soliton.

To derive the dark matter density, we follow the approach of Ref.[6]. For $\lambda = 0$, the field begins to oscillate at $\phi = F$ when the Hubble parameter satisfies $H \sim \frac{T_{osc}^2}{m_P} = m$, where $m_P = 1/\sqrt{8\pi G}$ is the reduced Planck mass, and $T_{osc} = \sqrt{mm_P}$ represents the temperature at that time. The typical energy density of ULDM at this time is of order of $F^2 m^2$. At the matter-radiation equality with the temperature $T_{eq} \simeq 1\text{eV}$ we find the relation $\frac{m^2 F^2}{T_{osc}^4} \frac{T_{osc}}{T_{eq}} \sim 1$.

Therefore,

$$F \sim \frac{m_P^{3/4} T_{eq}^{1/2}}{m^{1/4}} \sim 10^{17} \text{ GeV} \left(\frac{10^{-22} \text{ eV}}{m} \right)^{1/4}, \quad (32)$$

which indicates that the typical value of ϕ lies near the Grand Unified Theory (GUT) scale. From the present Hubble parameter H_0 and the temperature of the universe one can estimate the density parameter today for ULDM [6],

$$\Omega_\phi \sim 0.1 \left(\frac{F}{10^{17} \text{ GeV}} \right)^2 \left(\frac{m}{10^{-22} \text{ eV}} \right)^{1/2}. \quad (33)$$

For $\lambda \neq 0$, the above logic used for the $\lambda = 0$ case cannot be directly applied, as the equation of state for ϕ depends on the field value [58]. To treat the oscillation of ϕ as cold dark matter, the quartic term in $U(\phi)$ must be smaller than the quadratic term, at least at the time when the temperature is T_{eq} . That is, $\phi_{eq} < m/\sqrt{\lambda}$ must hold at that time. Considering oscillation of the field one can obtain the present density parameter

$$\Omega_\phi \simeq \frac{\frac{m^2 \phi_{eq}^2}{2} \left(\frac{T_{now}}{T_{eq}} \right)^3}{3H_0^2 m_P^2} \lesssim \frac{m^4}{6\lambda H_0^2 m_P^2} \left(\frac{T_{now}}{T_{eq}} \right)^3 \quad (34)$$

which leads to

$$\phi_{eq} = \sqrt{\frac{6\Omega_\phi H_0^2 m_P^2}{m^2} \left(\frac{T_{eq}}{T_{now}} \right)^3} \simeq 1.2 \times 10^{13} \text{ GeV} \left(\frac{10^{-22} \text{ eV}}{m} \right), \quad (35)$$

and [59]

$$\tilde{m} = m/\lambda^{1/4} \gtrsim \left(6\Omega_\phi \left(\frac{T_{eq}}{T_{now}} \right)^3 H_0^2 m_P^2 \right)^{1/4} \simeq 1.1 \text{ eV}, \quad (36)$$

where $T_{now} = 2.3 \times 10^{-4} \text{ eV}$ is the current temperature of the universe and $\Omega_\phi \simeq 0.26$.

On the other hand, the cross-section inferred from galaxy cluster collisions suggests $a_s^2/m \lesssim 1 \text{ cm}^2/g$. [43, 60, 61], which leads to

$$\lambda < 5.3 \times 10^{-44} \left(\frac{m}{10^{-22} \text{ eV}} \right)^{3/2}. \quad (37)$$

Taking all these into account, we expect $1 \text{ eV} \lesssim \tilde{m} \lesssim 10 \text{ eV}$ for $m \lesssim 10^{-5} \text{ eV}$. One can also obtain the field value at the onset of oscillations:

$$\phi_{osc} = \frac{\pi^{4/3} m_P \left(\frac{g_* T_{eq}}{\tilde{m}} \right)^{2/3}}{30^{2/3}} = 4.8 \times 10^{17} \text{ GeV} \left(\frac{\tilde{m}}{10 \text{ eV}} \right)^{-2/3}, \quad (38)$$

where $g_* = 3.36$ is the number of relativistic degrees of freedom. (See Ref. [46] for details.)

Fig. 2 illustrates the allowed parameter regions that satisfy these constraints. From the figure, cosmological constraints appear to favor the fiducial values ($m \sim 10^{-21} \text{ eV}$, $\lambda \sim 10^{-88}$), if we also incorporate constraints from the effective number of relativistic species during Big Bang nucleosynthesis ($10^{-19} \text{ eV} \lesssim m \lesssim 10^{-21} \text{ eV}$) [58].

V. RESULTS AND OUTLOOK

Through the analysis of the SPE derived from ϕ^4 scalar field theory, we have estimated the characteristic scales of physical quantities observed in galaxies for both the free ($\lambda = 0$) and self-interacting ($\lambda > 0$) ULDM scenarios. By accounting for both quantum pressure and self-interaction pressure, the limitations of each model can be addressed. For parameters consistent with observations, dark matter halos associated with galaxies are expected to begin collapsing at high redshifts ($z \gtrsim 100$) in the TF limit. Compared to λ , the mass m of self-interacting ULDM is subject to fewer known constraints, warranting further investigation.

The inferred density and typical field values of ULDM suggest a possible connection to GUT-scale physics. The existence of ULDM oscillations with a frequency $\sim m$ could be detectable through pulsar timing array experiments

[62], or atomic clock experiments [63]. Efforts to detect ULDM oscillations via precision frequency measurements rely on the premise that ULDM induces minute fluctuations in fundamental constants which, in turn, can be detected as variations in the frequencies of atomic clocks. The gravitational lensing [64] and the gravitational-wave signatures [65] can also offer a promising avenue to detect or constrain self-interacting ultralight dark matter. The characteristic scales derived in this work offer a convenient means of estimating key physical quantities of galaxies.

Future observations of early galaxies, including data from the James Webb Space Telescope [66], will provide new insights into galaxy evolution, potentially validating these characteristic scales. Exploring characteristic scales arising from ultralight axions with a cosine potential constitutes a promising direction for future research.

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